

The Breadth-one D -invariant Polynomial Subspace

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Abstract. We demonstrate the equivalence of two classes of D -invariant polynomial subspaces introduced in [8] and [9], i.e., these two classes of subspaces are different representations of the breadth-one D -invariant subspace. Moreover, we solve the discrete approximation problem in ideal interpolation for the breadth-one D -invariant subspace. Namely, we find the points, such that the limiting space of the evaluation functionals at these points is the functional space induced by the given D -invariant subspace, as the evaluation points all coalesce at one point.

Keywords. D -invariant polynomial subspace; Breadth-one; Ideal interpolation; Discrete approximation problem.

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1 Introduction

A polynomial subspace is said to be D -invariant if it is closed under differentiation. The breadth of a D -invariant polynomial subspace is defined as the number of all linear polynomials in a basis of this space. This number may not be unique when different bases are considered, here we choose the maximum as the breadth of the space. The breadth-one D -invariant subspace is mentioned by Dayton and Zeng in [3, 2], and is further discussed by Li and Zhi in [8] where a recursion formula of each polynomial in the basis is given. We provide another class of D -invariant subspaces in [9].

Ideal interpolation is originated in the paper of G. Birkhoff [1] where the interpolation problem is defined by a linear projector. C. de Boor and B. Shekhtman survey some results and raise some problems in their review articles [4] and [10], respectively. In ideal interpolation, the interpolation conditions at an interpolation site \mathbf{z} can be described by a space of linear functionals, i.e., $\text{span}\{\delta_{\mathbf{z}} \circ p(D), p \in P_{\mathbf{z}}\}$, where $P_{\mathbf{z}}$ is a D -invariant polynomial subspace, $\delta_{\mathbf{z}}$ is the evaluation functional at \mathbf{z} and $p(D)$ is the differential operator induced by p . *Lagrange interpolation* is a standard example where all $P_{\mathbf{z}} = \text{span}\{1\}$. Note that in one variable every ideal interpolation (over complex field) is the

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pointwise limit of Lagrange interpolation, and this is still true for some multivariate examples [5, 6, 7]. However, B. Shekhtman provides counterexamples when there are more than two variables [11]. In [12], B. Shekhtman prescribes finitely many computations to determine whether a projector is a limit of Lagrange projectors. But as pointed out by the author, “finitely many” is still far too many steps for a computer to handle even in a very simple case. This is the main motivation for us to consider the problem:

Given an ideal interpolant with its interpolation conditions $\delta_{\mathbf{z}}P_{\mathbf{z}}(D)$, where $P_{\mathbf{z}}$ is a D -invariant $n + 1$ -dimensional polynomial subspace, find $n + 1$ points $\mathbf{z}_0(h), \mathbf{z}_1(h), \dots, \mathbf{z}_n(h)$ such that

$$\lim_{h \rightarrow 0} \text{span}\{\delta_{\mathbf{z}_0(h)}, \delta_{\mathbf{z}_1(h)}, \dots, \delta_{\mathbf{z}_n(h)}\} = \{\delta_{\mathbf{z}} \circ p(D) : p \in P_{\mathbf{z}}\}. \quad (1)$$

We call this problem the *discrete approximation problem* for $\delta_{\mathbf{z}}P_{\mathbf{z}}(D)$ and $\mathbf{z}_0(h), \mathbf{z}_1(h), \dots, \mathbf{z}_n(h)$ the *discrete points* for $\delta_{\mathbf{z}}P_{\mathbf{z}}(D)$.

Actually, this question is first raised by C. de Boor and A. Ron in [7] where $P_{\mathbf{z}}$ is a D -invariant subspace spanned by homogeneous polynomials. We have solved this problem for the case that $P_{\mathbf{z}}$ is a D -invariant subspace with maximal total degree two in another paper. In this paper we will solve the problem for a particular case when $P_{\mathbf{z}}$ is a breadth-one D -invariant subspace.

The paper is organised as follows. We first prove that the two classes of D -invariant subspaces introduced in [8] and [9] are equivalent in the sense of coordinate transformation in Section 3. Then in Section 4, we solve the discrete approximation problem for the breadth-one D -invariant subspace in two ways. Namely, we present two sets of discrete points for this special subspace. The next section is devoted to introducing some notation and known results.

2 Preliminaries

Throughout the paper, \mathbb{F} denotes a field with characteristic zero. $\mathbb{F}[\mathbf{x}] := \mathbb{F}[x_1, \dots, x_d]$ denotes the polynomial ring in d variables over \mathbb{F} . For $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_+^d$, $\boldsymbol{\alpha}! := \alpha_1! \cdots \alpha_d!$. For $p = \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \in \mathbb{F}[\mathbf{x}]$, $p(D)$ is defined as

$$p(D) := \sum_{\boldsymbol{\alpha}} a_{\boldsymbol{\alpha}} \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

We define an operator Ψ_j on $\mathbb{F}[\mathbf{x}]$ that acts as “integral”:

$$\Psi_j(\mathbf{x}^{\boldsymbol{\alpha}}) := \frac{1}{\alpha_j + 1} x_1^{\alpha_1} \cdots x_j^{\alpha_j + 1} \cdots x_d^{\alpha_d}, \quad j = 1, \dots, d.$$

Li and Zhi demonstrate the structure of the breadth-one D -invariant polynomial subspace as follows:

Theorem 1 [8] *Let $\mathcal{L}_n := \text{span}\{L_0, L_1, \dots, L_n\}$ be the breadth-one D -invariant polynomial subspace, where $L_0 = 1$, $L_1 = x_1$. We can construct the k -th degree*

polynomial incrementally for k from 2 to n by the following formula:

$$L_k = V_k + a_{k,2}x_2 + \cdots + a_{k,d}x_d, \quad (2)$$

where V_k has no free parameters and is obtained from the computed basis $\{L_1, \dots, L_{k-1}\}$ by the following formula:

$$V_k = \Psi_1(M_1) + \Psi_2((M_2)_{i_1=0}) + \cdots + \Psi_d((M_d)_{i_1=i_2=\cdots=i_{d-1}=0}),$$

where

$$M_1 = L_{k-1}, \quad M_j = a_{2,j}L_{k-2} + \cdots + a_{k-1,j}L_1, \quad 2 \leq j \leq d.$$

Here $i_1 = \cdots = i_{j-1} = 0$ means that we only pick up terms which do not contain variables in x_1, \dots, x_{j-1} , and $a_{i,j} \in \mathbb{F}$ are known parameters appearing in L_i for $2 \leq i \leq k-1$, $2 \leq j \leq d$.

Note that 1 is always in a D -invariant subspace and the linear polynomial L_1 has the form x_1 with an appropriate linear coordinate transformation. Thus the hypotheses in Theorem 1 are reasonable and every breadth-one D -invariant subspace can be written in the above form with specified parameters $a_{i,j}$. For example, for $d = 2$,

$$\begin{aligned} L_1 &= x_1; \quad L_2 = \frac{1}{2!}x_1^2 + a_{2,2}x_2; \\ L_3 &= \frac{1}{3!}x_1^3 + a_{2,2}x_1x_2 + a_{3,2}x_2; \\ L_4 &= \frac{1}{4!}x_1^4 + \frac{1}{2!}a_{2,2}x_1^2x_2 + a_{3,2}x_1x_2 + \frac{1}{2!}a_{2,2}^2x_2^2 + a_{4,2}x_2; \\ &\dots \end{aligned}$$

To introduce another class of D -invariant polynomial subspaces discussed in [9], we first need some notation. Let $\mathbf{b} = (b_1, b_2, \dots, b_n) \in \mathbb{N}_+^n$, $n \geq 2$ satisfying

$$b_1 = 1, \quad b_n > \cdots > b_2 \geq 2,$$

and let

$$\mathbf{c}_1 = (c_{1,1}, c_{1,2}, \dots, c_{1,n}), \dots, \mathbf{c}_d = (c_{d,1}, c_{d,2}, \dots, c_{d,n}) \in \mathbb{F}^n,$$

where $c_{1,1}, \dots, c_{d,1}$ are not all zero. Construct a map $\tau : (\mathbb{N}^n)^d \rightarrow \mathbb{N}$ defined by

$$\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d) = \sum_{j=1}^n b_j \sum_{i=1}^d \gamma_{i,j},$$

with $\boldsymbol{\gamma}_i = (\gamma_{i,1}, \gamma_{i,2}, \dots, \gamma_{i,n}) \in \mathbb{N}^n$, $i = 1, \dots, d$.

Theorem 2 [9] Let $\mathbf{b} = (1, b_2, \dots, b_n)$, $\mathbf{c}_i = (c_{i,1}, c_{i,2}, \dots, c_{i,n})$, $i = 1, \dots, d$, and the map τ be as above. Let $q_{n,m}$ be a polynomial defined by

$$q_{n,m} = \sum_{\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d) = m} \frac{\mathbf{c}_1^{\boldsymbol{\gamma}_1} \cdots \mathbf{c}_d^{\boldsymbol{\gamma}_d}}{\boldsymbol{\gamma}_1! \cdots \boldsymbol{\gamma}_d!} x_1^{|\boldsymbol{\gamma}_1|} \cdots x_d^{|\boldsymbol{\gamma}_d|}, \quad m = 0, 1, 2, \dots, b_n, \quad (3)$$

with $\boldsymbol{\gamma}_i = (\gamma_{i,1}, \gamma_{i,2}, \dots, \gamma_{i,n}) \in \mathbb{N}^n$, $\mathbf{c}_i^{\boldsymbol{\gamma}_i} = \prod_{j=1}^n c_{i,j}^{\gamma_{i,j}}$. Then the linear space

$$Q = \text{span}\{q_{n,m} : m = 0, 1, 2, \dots, b_n\}$$

is a $(b_n + 1)$ -dimensional D -invariant polynomial subspace.

3 The Structure of the D -invariant Subspace: Case of Breadth One

In this section, we will prove the equivalence of the two classes of D -invariant subspaces introduced in the previous section.

Lemma 3 The breadth of the D -invariant subspace Q given by Theorem 2 is 1.

Proof. By the construction of $q_{n,m}$ and $b_1 = 1$, we know that $\deg(q_{n,m}) = m$, for all $m = 1, 2, \dots, b_n$. Namely, there is only one linear polynomial in the basis of Q , thus the lemma is proved. \square

We denote by $q_{n,m}^*$ the polynomial obtained by specifying

$$\begin{aligned} \mathbf{b} &= (b_1, b_2, \dots, b_n) = (1, 2, \dots, n), \\ \mathbf{c}_1 &= (c_{1,1}, c_{1,2}, \dots, c_{1,n}) = (1, 0, \dots, 0), \\ \mathbf{c}_s &= (c_{s,1}, c_{s,2}, \dots, c_{s,n}) = (0, a_{2,s}, a_{3,s}, \dots, a_{n,s}), \quad s = 2, \dots, d, \end{aligned}$$

in $q_{n,m}$, where $a_{i,s}$, $i = 2, \dots, n$, $s = 2, \dots, d$ are the parameters in (2). We define $0^0 = 1$ in this paper. Since $c_{1,2} = \dots = c_{1,n} = 0$, $c_{2,1} = \dots = c_{d,1} = 0$, it follows that $\gamma_{1,2}, \dots, \gamma_{1,n}$, $\gamma_{2,1}, \dots, \gamma_{d,1}$ must be zero, or the corresponding term in $q_{n,m}$ will be zero, thus

$$\begin{aligned} q_{n,m}^* &= \sum_{\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d) = m} \frac{c_{1,1}^{\gamma_{1,1}} c_{2,2}^{\gamma_{2,2}} \cdots c_{2,n}^{\gamma_{2,n}} \cdots c_{d,2}^{\gamma_{d,2}} \cdots c_{d,n}^{\gamma_{d,n}}}{\gamma_{1,1}! \gamma_{2,2}! \cdots \gamma_{2,n}! \cdots \gamma_{d,2}! \cdots \gamma_{d,n}!} x_1^{|\boldsymbol{\gamma}_1|} \cdots x_d^{|\boldsymbol{\gamma}_d|} \\ &= \sum_{\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d) = m} \frac{1^{\gamma_{1,1}} a_{2,2}^{\gamma_{2,2}} \cdots a_{n,2}^{\gamma_{2,n}} \cdots a_{2,d}^{\gamma_{d,2}} \cdots a_{n,d}^{\gamma_{d,n}}}{\gamma_{1,1}! \gamma_{2,2}! \cdots \gamma_{2,n}! \cdots \gamma_{d,2}! \cdots \gamma_{d,n}!} x_1^{|\boldsymbol{\gamma}_1|} \cdots x_d^{|\boldsymbol{\gamma}_d|}, \end{aligned}$$

where $\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d)$ can be written as

$$\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d) = \gamma_{1,1} + 2(\gamma_{2,2} + \cdots + \gamma_{d,2}) + \cdots + n(\gamma_{2,n} + \cdots + \gamma_{d,n}). \quad (4)$$

For $d = 2$, one can verify that

$$\begin{aligned} q_{n,0}^* &= 1; \quad q_{n,1}^* = x_1; \quad q_{n,2}^* = \frac{1}{2!}x_1^2 + a_{2,2}x_2; \\ q_{n,3}^* &= \frac{1}{3!}x_1^3 + a_{2,2}x_1x_2 + a_{3,2}x_2; \\ q_{n,4}^* &= \frac{1}{4!}x_1^4 + \frac{1}{2!}a_{2,2}x_1^2x_2 + \frac{1}{2!}a_{2,2}^2x_2^2 + a_{3,2}x_1x_2 + a_{4,2}x_2; \\ &\dots \end{aligned}$$

It is easy to see that for $d = 2$, $q_{n,i}^* = L_i, i = 1, \dots, 4$. More generally, we have

Proposition 4 *With the above notation, $\forall n \geq 0$,*

$$q_{n,m}^* = L_m, \quad m = 0, 1, \dots, n. \quad (5)$$

Proof. We will use induction on m . From the previous discussion $q_{n,0}^* = L_0, q_{n,1}^* = L_1$. Now assume that the proposition is true for all $m \leq n-1$. By Lemma 3.4 and Theorem 3.1 in [8], we know that

$$\begin{aligned} \frac{\partial L_n}{\partial x_1} &= L_{n-1}; \\ \frac{\partial L_n}{\partial x_j} &= a_{2,j}L_{n-2} + \dots + a_{n,j}L_0, \quad 2 \leq j \leq d. \end{aligned}$$

By the proof of Proposition 4.1 in [9], we know that for any $n \geq 2$,

$$\begin{aligned} \frac{\partial q_{n,n}^*}{\partial x_1} &= c_{1,1}q_{n,n-1}^* + \sum_{i=2}^n c_{1,i}q_{n,n-i}^* = q_{n,n-1}^*; \\ \frac{\partial q_{n,n}^*}{\partial x_j} &= c_{j,1}q_{n,n-1}^* + \sum_{i=2}^n c_{j,i}q_{n,n-i}^* = a_{2,j}q_{n,n-2}^* + \dots + a_{n,j}q_{n,0}^*, \quad 2 \leq j \leq d. \end{aligned}$$

Since $q_{n,m}^* = L_m$ for $0 \leq m \leq n-1$ by our inductive assumption, it follows that

$$\frac{\partial q_{n,n}^*}{\partial x_j} = \frac{\partial L_n}{\partial x_j}, \quad 1 \leq j \leq d.$$

Note that $q_{n,n}^*$ and L_n do not contain constant term, thus $q_{n,n}^* = L_n$. This completes the proof. \square

Theorem 5 *The subspace $Q = \text{span}\{q_{n,m} : m = 0, 1, 2, \dots, b_n\}$ given in Theorem 2 is equivalent to the breadth-one D -invariant subspace*

$$\mathcal{L}_{b_n} = \text{span}\{L_0, L_1, \dots, L_{b_n}\}$$

in the sense of coordinate transformation.

Proof. By Lemma 3, $Q \subset \text{span}\{L_0, L_1, \dots, L_{b_n}\}$ holds in the sense of coordinate transformation. By Proposition 4, we know that L_m can be obtained by specifying the parameters in $q_{n,m}$, thus the opposite inclusion holds and the theorem is proved. \square

Corollary 6 $L_k, k = 0, 1, \dots, n$, in Theorem 1 has the explicit expression:

$$L_k = \sum_{\tau(\gamma_1, \dots, \gamma_d)=k} \frac{1^{\gamma_{1,1}} a_{2,2}^{\gamma_{2,2}} \cdots a_{n,2}^{\gamma_{2,n}} \cdots a_{2,d}^{\gamma_{d,2}} \cdots a_{n,d}^{\gamma_{d,n}}}{\gamma_{1,1}! \gamma_{2,2}! \cdots \gamma_{2,n}! \cdots \gamma_{d,2}! \cdots \gamma_{d,n}!} x_1^{|\gamma_1|} \cdots x_d^{|\gamma_d|},$$

where τ is defined by (4).

If we think of L_0, L_1, \dots, L_n as a sequence of functions, then Theorem 1 describes the recursion formula of this sequence of functions while the above corollary provides the general formula.

4 The Discrete Approximation Problem for the Breadth-one D -invariant Subspace

In this section, we will solve the discrete approximation problem for the special class of D -invariant subspaces $\text{span}\{L_0, L_1, \dots, L_n\}$, which shows that an ideal interpolant, with its interpolation conditions of the form $\text{span}\{\delta_{\mathbf{z}} \circ L_0(D), \dots, \delta_{\mathbf{z}} \circ L_n(D)\}$ is a limit of Lagrange interpolants. The following lemma has been proved in [13], here we give a proof based on linear algebra.

Lemma 7 [13] For any nonnegative integers j and m ,

$$\sum_{i=0}^m (-1)^{m-i} \frac{1}{i!(m-i)!} i^j = \begin{cases} 1, & j = m; \\ 0, & 0 \leq j < m. \end{cases}$$

Proof. Consider the following linear equations:

$$\begin{pmatrix} 0^0 & 1^0 & \cdots & m^0 \\ 0^1 & 1^1 & \cdots & m^1 \\ \vdots & \vdots & & \vdots \\ 0^m & 1^m & \cdots & m^m \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

using Cramer's rules, we get

$$y_i = \frac{(-1)^{m-i}}{i!(m-i)!}, \quad \forall i = 0, 1, \dots, m.$$

Thus the lemma is proved. \square

Remove the first row and the first column of the coefficient matrix in the proof of the above lemma, we immediately obtain the following result.

Lemma 8 Let j, m be nonnegative integers and $1 \leq j \leq m$. Then

$$\sum_{i=1}^m (-1)^{m-i} \frac{1}{i!(m-i)!} i^j = \begin{cases} 1, & j = m; \\ 0, & 1 \leq j < m. \end{cases}$$

For any fixed $m > 0$, we define

$$A_k^{(m)} := (-1)^{m-k} \frac{1}{k!(m-k)!}, \quad k = 0, \dots, m.$$

Theorem 9 Let \mathbf{z}_0 be a base point and \mathcal{L}_n be as in Theorem 1. Define

$$\mathbf{z}_i(h) := \mathbf{z}_0 + (ih, \sum_{j=2}^n a_{j,2}(ih)^j, \dots, \sum_{j=2}^n a_{j,d}(ih)^j), \quad i = 0, \dots, n. \quad (6)$$

Then for any function f analytic at \mathbf{z}_0 ,

$$(L_m(D)f)(\mathbf{z}_0) = \lim_{h \rightarrow 0} \frac{1}{h^m} \left(\sum_{r=0}^m A_r^{(m)} f(\mathbf{z}_r(h)) \right), \quad \forall m = 0, \dots, n.$$

In other words, the points defined in (6) are discrete points satisfying

$$\text{span}\{\delta_{\mathbf{z}_0} \circ L_0(D), \dots, \delta_{\mathbf{z}_0} \circ L_n(D)\} = \lim_{h \rightarrow 0} \text{span}\{\delta_{\mathbf{z}_i(h)}, \quad i = 0, \dots, n\}. \quad (7)$$

Proof. By Taylor expansion and the definition of τ in (4), we have

$$\begin{aligned} f(\mathbf{z}_i(h)) &= \sum_{\Gamma=0}^{\infty} \left(ih \frac{\partial}{\partial x_1} + \sum_{j=2}^n a_{j,2}(ih)^j \frac{\partial}{\partial x_2} + \dots + \sum_{j=2}^n a_{j,d}(ih)^j \frac{\partial}{\partial x_d} \right)^{\Gamma} f(\mathbf{z}_0) \\ &= \sum_{|\boldsymbol{\gamma}_1| + \dots + |\boldsymbol{\gamma}_d| = 0}^{\infty} \frac{1}{|\boldsymbol{\gamma}_1|! \dots |\boldsymbol{\gamma}_d|!} (ih)^{|\boldsymbol{\gamma}_1|} \left(\sum_{j=2}^n a_{j,2}(ih)^j \right)^{|\boldsymbol{\gamma}_2|} \dots \left(\sum_{j=2}^n a_{j,d}(ih)^j \right)^{|\boldsymbol{\gamma}_d|} \frac{\partial^{|\boldsymbol{\gamma}_1| + \dots + |\boldsymbol{\gamma}_d|} f(\mathbf{z}_0)}{\partial x_1^{|\boldsymbol{\gamma}_1|} \dots \partial x_d^{|\boldsymbol{\gamma}_d|}} \\ &= \sum_{s=0}^m \sum_{\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d) = s} (ih)^s \frac{1^{\gamma_{1,1}} a_{2,2}^{\gamma_{2,2}} \dots a_{n,2}^{\gamma_{n,2}} \dots a_{2,d}^{\gamma_{d,2}} \dots a_{n,d}^{\gamma_{d,n}}}{\gamma_{1,1}! \gamma_{2,2}! \dots \gamma_{2,n}! \dots \gamma_{d,2}! \dots \gamma_{d,n}!} \frac{\partial^{|\boldsymbol{\gamma}_1| + \dots + |\boldsymbol{\gamma}_d|} f(\mathbf{z}_0)}{\partial x_1^{|\boldsymbol{\gamma}_1|} \dots \partial x_d^{|\boldsymbol{\gamma}_d|}} + O(h^{m+1}). \end{aligned}$$

Thus by Lemma 7, $\forall m = 0, \dots, n$, we know that

$$\begin{aligned} &\sum_{r=0}^m A_r^{(m)} f(\mathbf{z}_r(h)) \\ &= h^m \sum_{\tau(\boldsymbol{\gamma}_1, \dots, \boldsymbol{\gamma}_d) = m} \frac{1^{\gamma_{1,1}} a_{2,2}^{\gamma_{2,2}} \dots a_{n,2}^{\gamma_{n,2}} \dots a_{2,d}^{\gamma_{d,2}} \dots a_{n,d}^{\gamma_{d,n}}}{\gamma_{1,1}! \gamma_{2,2}! \dots \gamma_{2,n}! \dots \gamma_{d,2}! \dots \gamma_{d,n}!} \frac{\partial^{|\boldsymbol{\gamma}_1| + \dots + |\boldsymbol{\gamma}_d|} f(\mathbf{z}_0)}{\partial x_1^{|\boldsymbol{\gamma}_1|} \dots \partial x_d^{|\boldsymbol{\gamma}_d|}} + O(h^{m+1}) \\ &= h^m (L_m(D)f)(\mathbf{z}_0) + O(h^{m+1}). \end{aligned}$$

Thus the theorem is proved. \square

Finally, we will give another set of discrete points for $\text{span}\{L_0, L_1, \dots, L_n\}$.

Lemma 10 For any fixed integers $r \geq 1, i \geq 2$,

$$\begin{aligned} & \sum_{\alpha_1 + 2\alpha_{22} + \dots + i\alpha_{2i} = r} i^{\alpha_1} [i(i-1)]^{\alpha_{22}} [i(i-1)(i-2)]^{\alpha_{23}} \dots [i!]^{\alpha_{2i}} \\ &= \sum_{\alpha_1 + 2\alpha_{22} + \dots + r\alpha_{2r} = r} i^{\alpha_1} [i(i-1)]^{\alpha_{22}} \dots [i(i-1) \dots (i-(r-1))]^{\alpha_{2r}}. \end{aligned}$$

Proof. If $i > r$, then $\alpha_1 + 2\alpha_{22} + \dots + r\alpha_{2r} + \dots + i\alpha_{2i} = r$ yields $\alpha_{2,r+1} = \dots = \alpha_{2i} = 0$, thus the equation holds. If $i < r$, then

$$\begin{aligned} & i^{\alpha_1} [i(i-1)]^{\alpha_{22}} \dots [i(i-1) \dots (i-(r-1))]^{\alpha_{2r}} \\ &= i^{\alpha_1} [i(i-1)]^{\alpha_{22}} \dots [i!]^{\alpha_{2i}} [i! \cdot 0]^{\alpha_{2,i+1}} \dots [i! \cdot 0 \dots (i-(r-1))]^{\alpha_{2r}}. \end{aligned}$$

Since $0^k = 0, \forall k \neq 0$ and $0^0 = 1$, it follows that

$$i^{\alpha_1} [i(i-1)]^{\alpha_{22}} \dots [i(i-1) \dots (i-(r-1))]^{\alpha_{2r}} \neq 0$$

if and only if

$$\alpha_{2,i+1} = \dots = \alpha_{2r} = 0.$$

This completes the proof. \square

Theorem 11 Let the base point be \mathbf{z}_0 and $d = 2$, define

$$\begin{aligned} \mathbf{z}_0(h) &:= \mathbf{z}_0, \quad \mathbf{z}_1(h) := \mathbf{z}_0 + (h, 0), \\ \mathbf{z}_i(h) &:= \mathbf{z}_0 + (ih, \sum_{j=2}^i i(i-1) \dots (i-j+1) a_{j,2} h^j), \quad i = 2, \dots, n. \end{aligned}$$

Then the set of points $\{\mathbf{z}_0(h), \mathbf{z}_1(h), \dots, \mathbf{z}_n(h)\}$ also satisfies (7).

Proof. For an arbitrary f analytic at \mathbf{z}_0 , with Taylor expansion, we have

$$\begin{aligned} f(\mathbf{z}_1(h)) &= \sum_{\alpha_1=0}^{\infty} \frac{1}{\alpha_1!} \frac{\partial^{\alpha_1} f(\mathbf{z}_0)}{\partial x_1^{\alpha_1}} h^{\alpha_1}; \\ f(\mathbf{z}_i(h)) &= \sum_{k=0}^{\infty} \sum_{\alpha_1 + \alpha_2 = k} \frac{1}{\alpha_1! \alpha_2!} (ih)^{\alpha_1} \left(\sum_{j=2}^i i(i-1) \dots (i-j+1) a_{j,2} h^j \right)^{\alpha_2} \frac{\partial^k f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \\ &= \sum_{k=0}^{\infty} \sum_{\alpha_1 + \alpha_2 = k} \sum_{\alpha_{22} + \dots + \alpha_{2i} = \alpha_2} \frac{a_{22}^{\alpha_{22}} \dots a_{i2}^{\alpha_{2i}}}{\alpha_1! \alpha_{22}! \dots \alpha_{2i}!} i^{\alpha_1} (i(i-1))^{\alpha_{22}} \dots (i!)^{\alpha_{2i}} h^{\alpha_1 + 2\alpha_{22} + \dots + i\alpha_{2i}} \frac{\partial^k f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \\ &= \sum_{k=0}^{\infty} \sum_{\alpha_1 + \alpha_{22} + \dots + \alpha_{2i} = k} \frac{a_{22}^{\alpha_{22}} \dots a_{i2}^{\alpha_{2i}}}{\alpha_1! \alpha_{22}! \dots \alpha_{2i}!} i^{\alpha_1} (i(i-1))^{\alpha_{22}} \dots (i!)^{\alpha_{2i}} h^{\alpha_1 + 2\alpha_{22} + \dots + i\alpha_{2i}} \frac{\partial^k f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_{22} + \dots + \alpha_{2i}}} \end{aligned}$$

for $i = 2, \dots, n$. For convenience, we will write

$$i^{\alpha_1} (i(i-1))^{\alpha_{22}} \dots (i(i-1) \dots (i-(r-1)))^{\alpha_{2r}} \triangleq i^{\alpha_1 + 2\alpha_{22} + \dots + r\alpha_{2r}} + \omega(i),$$

where $\omega(i)$ is a polynomial in i with degree less than $\alpha_1 + 2\alpha_{22} + \dots + r\alpha_{2r}$.

For any fixed $m \in \{0, \dots, n\}$, consider the following combination

$$\sum_{i=0}^m A_i^{(m)} f(\mathbf{z}_i(h)) \triangleq \sum_{i=0}^m W_i h^i + O(h^{m+1}), \quad (8)$$

where

$$\begin{aligned} W_0 &= \sum_{i=0}^m A_i^{(m)}; \\ W_1 &= \sum_{i=1}^m A_i^{(m)} \sum_{\alpha_1=1} \frac{i^{\alpha_1}}{\alpha_1!} \frac{\partial^{\alpha_1} f(\mathbf{z}_0)}{\partial x_1^{\alpha_1}} = \sum_{i=1}^m A_i^{(m)} i \frac{\partial f(\mathbf{z}_0)}{\partial x_1}; \\ W_r &= \sum_{i=1}^m A_i^{(m)} \sum_{\alpha_1+2\alpha_{22}+\dots+i\alpha_{2i}=r} \frac{1}{\alpha_1! \alpha_{22}! \dots \alpha_{2i}!} i^{\alpha_1} (i(i-1))^{\alpha_{22}} \dots (i!)^{\alpha_{2i}} \frac{\partial^{\alpha_1+\alpha_{22}+\dots+\alpha_{2i}} f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_{22}+\dots+\alpha_{2i}}} \\ &= \sum_{i=1}^m A_i^{(m)} \sum_{\alpha_1+2\alpha_{22}+\dots+r\alpha_{2r}=r} \left(\frac{1}{\alpha_1! \alpha_{22}! \dots \alpha_{2r}!} \right. \\ &\quad \cdot i^{\alpha_1} (i(i-1))^{\alpha_{22}} \dots (i(i-1) \dots (i-(r-1)))^{\alpha_{2r}} \frac{\partial^{\alpha_1+\alpha_{22}+\dots+\alpha_{2r}} f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_{22}+\dots+\alpha_{2r}}} \Big) \\ &= \sum_{i=1}^m A_i^{(m)} \sum_{\alpha_1+2\alpha_{22}+\dots+r\alpha_{2r}=r} \frac{1}{\alpha_1! \alpha_{22}! \dots \alpha_{2r}!} (i^r + \omega(i)) \frac{\partial^{\alpha_1+\alpha_{22}+\dots+\alpha_{2r}} f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_{22}+\dots+\alpha_{2r}}} \\ &= \sum_{\alpha_1+2\alpha_{22}+\dots+r\alpha_{2r}=r} \frac{1}{\alpha_1! \alpha_{22}! \dots \alpha_{2r}!} \sum_{i=1}^m A_i^{(m)} (i^r + \omega(i)) \frac{\partial^{\alpha_1+\alpha_{22}+\dots+\alpha_{2r}} f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_{22}+\dots+\alpha_{2r}}} \end{aligned}$$

for $2 \leq r \leq m$, here we define $\alpha_{21} = 0$. The second equation in W_r holds according to Lemma 10.

By Corollary 6 and Lemma 7, 8, we know that

$$\begin{aligned} &\sum_{i=0}^m A_i^{(m)} f(\mathbf{z}_i(h)) \\ &= h^m \sum_{\alpha_1+2\alpha_{22}+\dots+m\alpha_{2m}=m} \frac{1}{\alpha_1! \alpha_{22}! \dots \alpha_{2m}!} \frac{\partial^{\alpha_1+\alpha_{22}+\dots+\alpha_{2m}} f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_{22}+\dots+\alpha_{2m}}} + O(h^{m+1}) \\ &= h^m \sum_{\alpha_1+2\alpha_{22}+\dots+n\alpha_{2n}=m} \frac{1}{\alpha_1! \alpha_{22}! \dots \alpha_{2n}!} \frac{\partial^{\alpha_1+\alpha_{22}+\dots+\alpha_{2n}} f(\mathbf{z}_0)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_{22}+\dots+\alpha_{2n}}} + O(h^{m+1}) \\ &= h^m (L_m(D)f)(\mathbf{z}_0) + O(h^{m+1}), \end{aligned}$$

thus

$$\lim_{h \rightarrow 0} \frac{1}{h^m} \left(\sum_{i=0}^m A_i^{(m)} f(\mathbf{z}_i(h)) \right) = (L_m(D)f)(\mathbf{z}_0), \quad \forall m = 2, \dots, n,$$

that is,

$$\delta_{\mathbf{z}_0} \circ L_m(D) = \lim_{h \rightarrow 0} \frac{1}{h^m} \left(\sum_{i=0}^m A_i^{(m)} \delta_{\mathbf{z}_i(h)} \right).$$

It is easy to verify that the above equation also holds for $m = 1$. Thus the theorem is proved. \square

For $d \geq 3$, we give the following result without a proof.

Theorem 12 *Suppose that the base point is \mathbf{z}_0 , define $\mathbf{z}_0(h) := \mathbf{z}_0$, $\mathbf{z}_1(h) := \mathbf{z}_0 + (h, 0, \dots, 0)$ and for $i = 2, \dots, n$,*

$$\mathbf{z}_i(h) := \mathbf{z}_0 + (ih, \sum_{j=2}^i \frac{i!}{(i-j)!} a_{j,2} h^j, \sum_{j=2}^i \frac{i!}{(i-j)!} a_{j,3} h^j, \dots, \sum_{j=2}^i \frac{i!}{(i-j)!} a_{j,d} h^j).$$

Then $\{\mathbf{z}_0(h), \mathbf{z}_1(h), \dots, \mathbf{z}_n(h)\}$ is a set of discrete points for the breadth-one subspace $\mathcal{L}_n = \text{span}\{L_0, L_1, \dots, L_n\}$.

Example 1. Let $d = 2, n = 4, a_{2,2} = 2, a_{3,2} = 3, a_{4,2} = 4$, thus,

$$\mathcal{L}_4 = \text{span}\{1, x_1, \frac{1}{2}x_1^2 + 2x_2, \frac{1}{3!}x_1^3 + 2x_1x_2 + 3x_2, \frac{1}{4!}x_1^4 + x_1^2x_2 + 3x_1x_2 + 2x_2^2 + 4x_2\}.$$

Let $\mathbf{z}_0 := (0, 0)$, then by Theorem 9 and Theorem 11,

$$\{(0, 0), (h, 0), (2h, 2(2h)^2 + 3(2h)^3 + 4(2h)^4), \\ (3h, 2(3h)^2 + 3(3h)^3 + 4(3h)^4), (4h, 2(4h)^2 + 3(4h)^3 + 4(4h)^4)\}$$

and

$$\{(0, 0), (h, 0), (2h, 4h^2), \\ (3h, 12h^2 + 18h^3), (4h, 24h^2 + 72h^3 + 96h^4)\}$$

are two sets of discrete points satisfying (7).

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